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A comparison of the entanglement measures negativity and concurrence

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Abstract
In this paper we investigate two different entanglement measures in the case of mixed states of two qubits. We prove that the negativity of a state can never exceed its concurrence and is always larger than $\sqrt{(1-C)^2+C^2}-(1-C)$, where C is the concurrence of the state. Furthermore, we derive an explicit expression for the states for which the upper or lower bound is satisfied. Finally we show that similar results hold if the relative entropy of entanglement and the entanglement of formation are compared.

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The concept of negativity originates from the observation by Peres [1] that a partial transpose of a density matrix associated with a separable state is still a valid density matrix and thus positive (semi)definite. Subsequently, Horodecki *et al* [2] proved that this was a necessary and sufficient condition for a state to be separable if the dimension of the Hilbert space does not exceed 6. The criterion of Peres leads to a natural entanglement measure called the negativity N, defined as

$$N(\rho) = \max(0, -2\lambda_{\min}) \tag{1}$$

where $\lambda_{\min}$ is the smallest eigenvalue of the partial transpose of the state ρ. Recently, Vidal and Werner proved that the negativity is an entanglement monotone and therefore a good entanglement measure [3]. Furthermore, the concept of negativity is of importance as it leads to upper bounds for the entanglement of distillation.

The concept of concurrence originates from the seminal work of Hill and Wootters [4, 5] where the exact expression of the entanglement of formation of a system of two qubits was derived. They showed that the entanglement of formation, an entropic entanglement monotone, is a convex monotonic increasing function of the concurrence.

Both measures have the same dimensionality and it is, therefore, natural to compare them, as one is related to the concept of entanglement of formation and the other one to the concept of entanglement of distillation.

We will derive the possible range of values for the negativity if the concurrence of the state is known. First of all we prove the following conjecture by Eisert *et al* [8] and Życzkowski [9]:

Theorem 1. *The negativity of an entangled mixed state of two qubits can never exceed its concurrence.*

Proof. To prove this, we need the result of Wootters [5] that a state with a given concurrence can always be decomposed as a convex sum of four pure states all having the same concurrence. It is readily checked that the negativity of a pure state is exactly equal to its concurrence. Due to linearity of the partial trace operation, the negativity of a mixed state is now obtained by calculating the smallest eigenvalue of the matrix obtained by making the convex sum of the partial transposes of the four pure states all having an equal negative eigenvalue. It is a well-known result by Weyl that the minimal eigenvalue of the sum of matrices always exceeds the sum of the minimal eigenvalues, which concludes the proof. □

The next step is to find the lowest possible value of the negativity for given concurrence. To this end we need a parametrization of the manifold of states with constant concurrence. In [11], it was shown how the concurrence changes under the application of an LQCC operation of the type

$$\rho' = \frac{(A\otimes B)\rho(A\otimes B)^\dagger}{\mathrm{Tr}((A\otimes B)\rho(A\otimes B)^\dagger)}. \tag{2}$$

The transformation rule is

$$C(\rho') = C(\rho)\frac{|\det A||\det B|}{\mathrm{Tr}((A\otimes B)\rho(A\otimes B)^\dagger)}. \tag{3}$$

It was further shown that for each density matrix ρ there exist A and B such that ρ' is Bell diagonal. The concurrence of a Bell diagonal state is only dependent on its largest eigenvalue λ_1 [4]: $C(\rho_{BD}) = 2\lambda_1(\rho_{BD}) - 1$. It is then straightforward to obtain the parametrization of the surface of constant concurrence (and hence constant entanglement of formation): it consists of applying all complex full rank 2×2 matrices A and B on all Bell diagonal states with the given concurrence, under the constraint that

$$\mathrm{Tr}\left(\left(\frac{A^\dagger A}{|\det(A)|}\otimes\frac{B^\dagger B}{|\det B|}\right)\rho\right) = 1.$$

It is clear that we can restrict ourselves to matrices A and B having determinant 1 ($A, B \in SL(2, C)$), as will be done in the sequel.

The extremal values of the negativity can now be obtained in two steps: first finding the state with extremal negativity for given eigenvalues of the corresponding Bell diagonal state by varying A and B, and then optimizing over all Bell diagonal states with equal λ_1.

The first step can be achieved by differentiating the following cost function over the manifold of $A, B \in SL(2, C)$:

$$\begin{aligned}\Phi(A, B) &= \lambda_{\min}\left(((A\otimes B)\rho_{BD}(A\otimes B)^\dagger)^\Gamma\right) &(4)\\ &= \lambda_{\min}\left((A\otimes B^*)\rho_{BD}^\Gamma(A\otimes B^*)^\dagger\right) &(5)\end{aligned}$$

under the constraint

$$\mathrm{Tr}\left((A\otimes B^*)\rho_{BD}^\Gamma(A\otimes B^*)^\dagger\right) = 1$$

where the notation Γ is used to denote partial transposition.

There exists a very elegant formalism for differentiating the eigenvalues of a matrix: given the eigenvalue decomposition of a Hermitian matrix $X = U\Lambda U^\dagger$, it is easy to prove that $\dot\Lambda = \mathrm{diag}(U^\dagger \dot X U)$, where 'diag' means the diagonal elements of a matrix. We can readily apply this to our Lagrange constrained problem. Indeed, the complete manifold of interest is generated by varying A and B as $\dot A = KA$ and $\dot B = LB$ with K, L arbitrary complex 2×2 traceless matrices (the trace condition is necessary to keep the determinants constant). Moreover, the minimal eigenvalue is given by $\mathrm{Tr}(\mathrm{diag}[0;0;0;1]D)$ where D is the diagonal matrix containing the ordered eigenvalues of $C = PDP^\dagger = (A\otimes B^*)\rho_{BD}^\Gamma(A\otimes B^*)^\dagger$ and P, the eigenvectors of C. We proceed as

$$\dot\Phi = \mathrm{Tr}\left(P^\dagger \dot C P\left(\underbrace{\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&1\end{pmatrix} - \mu I_4}_{=J(\mu)}\right)\right)$$

$$\dot C = ((K\otimes I_2) + (I_2\otimes L))C + C((K^\dagger\otimes I_2) + (I_2\otimes L^\dagger))$$

where μ is the Lagrange multiplier. An extremum is obtained if $\dot\Phi$ vanishes for all possible traceless K and L. Some straightforward algebra shows that this condition is fulfilled iff $CPJ(\mu)P^\dagger = P(DJ(\mu))P^\dagger$ is Bell diagonal (up to local unitary transformations).

Next we have to distinguish two cases, namely when the Lagrange multiplier $\mu = 0$ and when $\mu \neq 0$. The first case leads to the condition that the eigenvector of ρ^Γ corresponding to the negative eigenvalue is a Bell state. It is indeed easily checked that all density matrices with this property have negativity equal to the concurrence, and this is clearly an extremal case. We have therefore identified the class of states for which the negativity is equal to the concurrence. It is interesting to note that all the pure states and all the Bell diagonal states belong to this class.

The problem becomes much more subtle when the Lagrange multiplier does not vanish. Using the arguments of the proof of theorem 5 in [11], it is easy to prove that the partial transpose of an entangled state is always full rank and has at most one negative eigenvalue: the set of equations (10)–(13) in [11] is inconsistent with the constraints $\lambda_3 \leqslant 0$ and $\lambda_4 < 0$. $P(DJ(\mu))P^\dagger$ will therefore be Bell diagonal either if the eigenvectors of C are Bell states, or possibly if $DJ(\mu)$ contains eigenvalues with a multiplicity of 2. In this last case the two eigenvectors corresponding to the multiple eigenvalue are not uniquely defined and can be rotated to Bell states if the two other eigenvectors were already Bell states. As the first case is already discussed in the previous paragraph, we concentrate on the second case. Denoting the eigenvalues of C as λ_1, λ_2, $\lambda_3 \geqslant 0 \geqslant \lambda_4$, the eigenvector corresponding to λ_4 can be different from a Bell state iff we choose the Lagrange multiplier such that $-\mu\lambda_3 = (1-\mu)\lambda_4$. The eigenvectors corresponding to λ_1 and λ_2 have to be Bell states. Therefore all states for which the eigenvectors of the partial transposes are, up to local unitary transformations, of the form

$$P = \begin{pmatrix}1/\sqrt2 & 1/\sqrt2 & 0 & 0\\ 0&0&1&0\\ 0&0&0&1\\ 1/\sqrt2 & -1/\sqrt2 & 0 & 0\end{pmatrix}\begin{pmatrix}I_2 & 0\\ 0 & U_2\end{pmatrix} \tag{6}$$

with U_2, an arbitrary 2×2 unitary matrix will give extremal values of the negativity. The next step is, therefore, to find the state belonging to this class with minimal negativity for fixed concurrence or equivalently, the one with the largest concurrence for fixed negativity. Parametrizing the unitary U as $\begin{pmatrix} a & -b \\ b^* & a^* \end{pmatrix}$, the class of states we are considering is parametrized as

$$\begin{pmatrix} \frac{\lambda_1+\lambda_2}{2} & 0 & 0 & ab(\lambda_3-\lambda_4) \\ 0 & \lambda_3|a|^2+\lambda_4|b|^2 & \frac{\lambda_1-\lambda_2}{2} & 0 \\ 0 & \frac{\lambda_1-\lambda_2}{2} & \lambda_3|b|^2+\lambda_4|a|^2 & 0 \\ a^*b^*(\lambda_3-\lambda_4) & 0 & 0 & \frac{\lambda_1+\lambda_2}{2} \end{pmatrix}.$$

The concurrence of this state can be calculated by finding the Cholesky decomposition of $\rho = XX^\dagger$ and calculating the singular values of $X^T(\sigma_y \otimes \sigma_y)X$. As ρ is a direct sum of two 2×2 matrices, this can be done exactly as

$$\sigma_1 = \frac{\lambda_1+\lambda_2}{2} + |ab|(\lambda_3-\lambda_4) \tag{7}$$

$$\sigma_3 = \frac{\lambda_1+\lambda_2}{2} - |ab|(\lambda_3-\lambda_4) \tag{8}$$

$$\sigma_2 = \sqrt{\left(\lambda_3|a|^2+\lambda_4|b|^2\right)\left(\lambda_3|b|^2+\lambda_4|a|^2\right)} + \frac{\lambda_1-\lambda_2}{2} \tag{9}$$

$$\sigma_4 = \sqrt{\left(\lambda_3|a|^2+\lambda_4|b|^2\right)\left(\lambda_3|b|^2+\lambda_4|a|^2\right)} - \frac{\lambda_1-\lambda_2}{2}. \tag{10}$$

The concurrence is, therefore, given by

$$C = 2(\lambda_3-\lambda_4)|ab| - 2\sqrt{\left(\lambda_3|a|^2+\lambda_4|b|^2\right)\left(\lambda_3|b|^2+\lambda_4|a|^2\right)}. \tag{11}$$

The task is now reduced to finding a, b, λ_1, λ_2, λ_3, such that C is maximized for fixed λ_4. Some long but straightforward calculations lead to the optimal solution

$$|a|^2 = 1-|b|^2 = \frac{\lambda_3}{|\lambda_4|} \tag{12}$$

$$\lambda_1 = \lambda_2 = \sqrt{\lambda_3|\lambda_4|} \tag{13}$$

$$1 = \lambda_1+\lambda_2+\lambda_3+\lambda_4. \tag{14}$$

This solution corresponds to a state with two vanishing eigenvalues, while the remaining two eigenvectors are a Bell state and a separable state orthogonal to it:

$$\rho = \begin{pmatrix} C/2 & 0 & 0 & C/2 \\ 0 & 1-C & 0 & 0 \\ 0 & 0 & 0 & 0 \\ C/2 & 0 & 0 & C/2 \end{pmatrix}. \tag{15}$$

The concurrence C is then related to the negativity $N = 2|\lambda_4|$ by the equation

$$N^2 + 2N(1-C) - C^2 = 0. \tag{16}$$

This equation defines the lower bound we were looking for, as it relates the minimal possible value of the negativity for given concurrence. The state for which this minimum is reached is

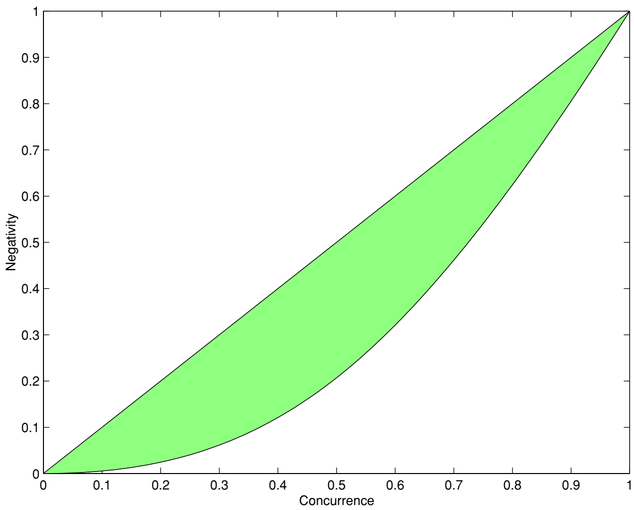

Figure 1. Range of values of the negativity for given concurrence.

special in the sense that it is a maximally entangled mixed state [7], [10]; no global unitary transformation can increase its entanglement. Moreover, it is the only mixed state that can be brought arbitrarily close to a Bell state by doing local operations (LOCC) on one copy of the state only; it is a quasi-distillable state [7]. We have therefore proven:

Theorem 2. *The negativity N of a mixed state with given concurrence C is always smaller than C with equality iff the eigenvector of ρ^Γ corresponding to its negative eigenvalue is a Bell state (up to local unitary transformations). Moreover, the negativity is always larger than $\sqrt{(1-C)^2+C^2}-(1-C)$, with equality iff the state is a rank 2 quasi-distillable state.*

A scatter plot of the negativity versus the concurrence for all entangled states is shown in figure 1.

A similar analysis can be performed to compare the entanglement of formation [5] and the relative entropy of entanglement [12]. It is well known that they coincide for pure states, and that the relative entropy of entanglement can never exceed the entanglement of formation. Due to the logarithmic nature of these quantities however, finding the states with minimal relative entropy of entanglement for given entanglement of formation is very hard to do analytically. Numerical investigations however showed that again the same quasi-distillable rank 2 states minimize the relative entropy of entanglement. It is indeed possible to show that these states are local minima to the optimization problem. Using the results of Verstraete *et al* [7], this minimal value is then given by

$$E_R(\rho) = (C-2)\log(1-C/2) + (1-C)\log(1-C). \tag{17}$$

A scatter plot of the range of values of the relative entropy of entanglement is given in figure 2.

Both the relative entropy of entanglement and the negativity lead to upper bounds on the entanglement of distillation. The strict lower bounds for these quantities, derived in this paper, are therefore nice illustrations of the expected irreversibility of entanglement manipulations in mixed states.

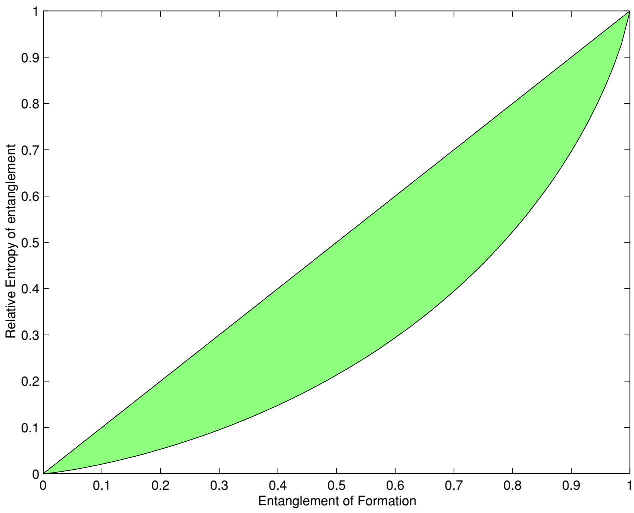

Figure 2. Range of values of the relative entropy of entanglement for given entanglement of formation.

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